Topological spaces

Future Mathematicians Programme - Mathematical structures exploration
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Idea. *Topological spaces* capture the idea of space. A topological space is a set equipped with a concept of proximity. It captures the idea that some points in a space are close to each other. An important class of examples is given by sets equipped with a notion of distance between elements, as we can then say that two elements are close to each other when the distance between them is small. But topological spaces can be way more general than this and arise everywhere in mathematics.

Definition. A ***topological space*** is a set X equipped with a collection of subsets of X, called *open subsets*, such that all the following properties hold:

(1) *empty, whole:* the empty subset $\emptyset$ and the whole set X are open;

(2) *arbitrary unions:* given a possibly infinite family of open subsets $\{U_i\}_{i\in I}$, their union $\bigcup_{i\in I} U_i$ is open;

(3) *finite intersections:* given open subsets U and V, their intersection $U \cap V$ is open (as a consequence the intersection of a finite number of open subsets is open).

The collection τ of the open subsets is called ***a topology*** on X.

A ***closed subset*** is a subset F of X that is the complement of an open subset U: that is, $F = X \setminus U$.

Remark. A topology captures the idea that some elements of the set are close to each other while some elements are further apart from each other. The idea is that the more open sets contain two points the closer the two points are to each other.

Problem 1. Consider the set $X = \{1, 2, 3, 4\}$. Show that

$$\tau = \{\emptyset, \{1\}, \{1, 2\}, \{3, 4\}, \{1, 3, 4\}, X\}$$

is a topology on X. We can think that this topology is saying that 3 and 4 are very close to each other, as every open subset containing 3 also contains 4 and viceversa. 2 and 4 are instead very far apart from each other, as the only open subset containing both of them is the whole X.

What are the closed subsets of X?

Definition. A ***base*** β ***for a topology*** is a collection of subsets of X such that all the following properties hold:

(a) the whole set X is the union of a possibly infinite family of subsets belonging to β: that is, $X = \bigcup_{i\in I} B_i$ with $B_i \in \beta$ for every i;

(b) given $B, C \in \beta$ and $x \in B \cap C$, there exists $D \in \beta$ such that $x \in D$ and $D \subseteq B \cap C$.

Remark. Every base β for a topology generates a topology by taking the open subsets to be either the empty set $\emptyset$ or possibly infinite unions of subsets belonging to the base β.

Problem 2. Consider the set $\mathbb{R}$ of real numbers. Show that the open intervals (x, y) with $x, y \in \mathbb{R}$ form a base for a topology. What are the closed subsets?

Consider now the plane $\mathbb{R}^2$ equipped with the euclidean distance $d(x, y) = \sqrt{x^2 + y^2}$. Can you construct a base B for a topology on $\mathbb{R}^2$ using the euclidean distance? Remember that open subsets (and thus in particular the subsets belonging to the base) should express that some points are close to each other.

Definition. A topological space (X, τ) is ***Hausdorff (or T2)*** if for every two points $x, y \in X$ with $x \neq y$ we can find two open subsets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. This means that we can always separate two points from each other.

Problem 3. Is the topological space of Problem 1 Hausdorff?

What about $\mathbb{R}$ and $\mathbb{R}^2$ with the topologies of Problem 2? Are they Hausdorff?

Consider now the set $\mathbb{N}$ of natural numbers with the following topology: a subset U of $\mathbb{N}$ is open precisely when $\mathbb{N} \setminus U$ is finite or $U = \emptyset$. Show that this is indeed a topology.

Is this topological space Hausdorff?

Interesting links to other mathematical structures.

It is sometimes very hard to study complex topological spaces. A good method is to associate some ***groups*** to topological spaces and analyze them.

Topological spaces are generalized by particular ***partially ordered sets*** equipped with meets and joins; these are called frames.